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EXTENSION OF OPERATORS WITH SEPARABLE RANGE

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ABSTRACT. A Banach space E is *injective* if it satisfies the following *extension property*: for every space X and every subspace Y of X , each operator $T : Y \rightarrow E$ admits an extension $\hat{T} : X \rightarrow E$. Many people have investigated these spaces, but it remains unknown whether every injective Banach space is isomorphic to a space of continuous functions $C(K)$ with K a Stonian compact.

We consider two weaker forms of injectivity: E is *separably injective* if it satisfies the extension property when X is separable; it is *universally separably injective* if it satisfies the extension property when Y is separable. Obviously, injective \Rightarrow universally separably injective \Rightarrow separably injective, but the converse implications fail. We show that the corresponding classes of Banach spaces are much richer in examples and structural properties than injectivity.

1. INTRODUCTION

A Banach space E is said to be λ -*injective* ($\lambda \geq 1$) if it satisfies the following *extension property*:

for every Banach space X and every subspace Y of X , each operator $T : Y \rightarrow E$ admits an extension $\hat{T} : X \rightarrow E$ satisfying $\|\hat{T}\| \leq \lambda\|T\|$.

The space E is *injective* if it is λ -injective for some $\lambda \geq 1$.

Nachbin, Goodner, Kelley and Hasumi [21, 10, 15, 12] characterized the 1-injective spaces as those Banach spaces linearly isometrically isomorphic to a $C(K)$ space, with K a Stonian compact. However, despite the deep investigations of Argyros [1, 2, 3, 4], Haydon [13], Rosenthal [23, 24] and other authors, finding a description of the class of injective Banach spaces seems to be an unmanageable problem. It is not even known if every injective space is isomorphic to a 1-injective space or to a $C(K)$ space.

We deal with two weaker forms of injectivity which admit a similar definition. Namely, we say that E is λ -*separably injective* (and write $E \in \Upsilon_\lambda$) if it satisfies the previously described extension property, but only for X separable. Also, we say that E is λ -*universally separably injective* (and write $E \in \Upsilon_\lambda^{univ}$) if it satisfies the extension property when Y is separable. Obviously, $\Upsilon_\lambda^{univ} \subset \Upsilon_\lambda$.

The space E is *separably injective* if $E \in \Upsilon := \bigcup_{\lambda \geq 1} \Upsilon_\lambda$; and it is *universally separably injective* if $E \in \Upsilon^{univ} := \bigcup_{\lambda \geq 1} \Upsilon_\lambda^{univ}$.

Our aim is to show that the classes Υ and Υ^{univ} are much richer in examples and structure than the class of injective spaces. Here we include only a few proofs. For a detailed account, we refer to [5].

Among other results, we give several characterizations of spaces in Υ_λ and prove some basic properties of these spaces: they are $\mathcal{L}_{\infty, \lambda+}$ -spaces and have Pełczyński's property (V). The space c_0 is in Υ_2 and every (infinite dimensional) separable Banach space in Υ is isomorphic to c_0 . A $C(K)$ space belongs to Υ_1 if and only if the compact

K is an F -space; we also give some other characterizations. We also show some stability properties of the classes Υ and Υ^{univ} : both have the three-space property; and if $(E_n) \subset \Upsilon_\lambda$, then $c_0(E_n) \in \Upsilon$. Moreover, if Y is a subspace of X , then $Y, X \in \Upsilon$ implies $X/Y \in \Upsilon$, and $X \in \Upsilon^{univ}$ and $Y \in \Upsilon$ imply $X/Y \in \Upsilon^{univ}$; thus $\ell_\infty/c_0 \in \Upsilon^{univ}$. The stability properties allow us to construct many new examples of spaces in Υ or Υ^{univ} . Among them, we show spaces in Υ^{univ} which are not isomorphic to any complemented subspace of any $C(K)$ space. We also show that an ultraproduct of Banach spaces (following a countably incomplete ultrafilter) is injective only in the trivial case in which it is finite dimensional. However, if an ultraproduct is a \mathcal{L}_∞ space, then it is universally separably injective.

Notations and Conventions. Throughout the paper the ground field is \mathbb{R} . Of course, most of our results can be adapted to the complex setting. The Banach-Mazur distance between the Banach spaces X and Y is

$$\text{dist}_{BM}(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism between } X \text{ and } Y\}.$$

A Banach space X is a $\mathcal{L}_{\infty, \lambda}$ -space (with $1 \leq \lambda < \infty$) if every finite dimensional subspace F of X is contained in another finite dimensional subspace of X whose Banach-Mazur distance to the corresponding ℓ_∞^n is at most λ . A space X is a \mathcal{L}_∞ -space if it is a $\mathcal{L}_{\infty, \lambda}$ -space for some $\lambda \geq 1$; and it is a $\mathcal{L}_{\infty, \lambda+}$ -space when it is a $\mathcal{L}_{\infty, \lambda'}$ -space for all $\lambda' > \lambda$.

We write $C(K)$ for the Banach space of all continuous functions on the compact space K , with the sup norm. Topological spaces are assumed to be Hausdorff. We write $|S|$ for the cardinality of a set S .

Let Γ be a set. We denote by $\ell_\infty(\Gamma)$ the space of all bounded scalar functions on Γ , endowed with the sup norm. Moreover, $c_0(\Gamma)$ is the closed subspace spanned by the characteristic functions of the singletons of Γ .

The *density character* $\text{dens}(X)$ of a Banach space X is the least cardinal \mathfrak{m} for which X has a dense subset of cardinality \mathfrak{m} . Observe that $\text{dens}(\ell_\infty(\Gamma)) = 2^{|\Gamma|}$.

This paper describes joint work with Antonio Avilés, Félix Cabello, Jesús M.F. Castillo and Yolanda Moreno [5]. It was presented during the R.I.M.S. Conference *Prospects of non-commutative analysis in operator theory* at Kyoto University, October 28-30, 2009. The author thanks Professors Muneo Chō and Kotaro Tanahashi for their attentions during this Conference.

2. INJECTIVE SPACES

A Banach space E is *injective* if for every Banach space X and every subspace Y of X , each operator $T: Y \rightarrow E$ admits an extension $\hat{T}: X \rightarrow E$.

The space E is *1-injective* if we can always get \hat{T} with $\|\hat{T}\| = \|T\|$.

Remark 1. *It is not difficult to show that E is injective if and only if every subspace of a Banach space isomorphic to E is complemented.*

The first examples of injective Banach spaces are obtained as a direct consequence of the Hahn-Banach Theorem:

- (1) \mathbb{K} is 1-injective.
- (2) $\ell_\infty(I)$ is 1-injective. Indeed, for each $T: Y \rightarrow \ell_\infty(I)$ there exists a family $(y_i^*)_{i \in I} \subset Y^*$ so that $Ty = (y_i^*(y))$ and $\|T\| = \sup_{i \in I} \|y_i^*\|$.

Remark 2. Every Banach space X can be embedded as a subspace of a $\ell_\infty(\Gamma)$ space with $|\Gamma| = \text{dens}(X)$. Indeed, let $\{x_i : i \in \Gamma\}$ be a dense subset of X and choose, for each $i \in \Gamma$, a norm-one $f_i \in X^*$ such that $f_i(x_i) = \|x_i\|$. Then the operator $T : X \longrightarrow \ell_\infty(\Gamma)$ defined by $T(x) := (f_i(x))$ is an isometric embedding.

As a consequence, a Banach space E is injective if and only if it is isomorphic to a complemented subspace of $\ell_\infty(\Gamma)$ for some set Γ .

In the period 1950–58, a characterization of the 1-injective Banach spaces was obtained in several steps by Nachbin, Goodner, Kelley and Hasumi. Recall that a topological space is said to be *Stonian* if the closure of each open subset is open.

Theorem 3. [21, 10, 15, 12]

Every 1-injective space is isometrically isomorphic to some $C(K)$ space, where K is a Stonian compact.

However, the following problems have remained open:

- (1) Is every injective space isomorphic to a 1-injective space?
- (2) Is every injective space isomorphic to a $C(K)$ space?
- (3) Which is the structure of an injective space?

Other examples of 1-injective spaces.

- (1) Let I be a non-empty set endowed with the discrete topology. Denoting by βI the Stone-Čech compactification of I , $C(\beta I) \equiv \ell_\infty(I) \equiv c_0(I)^{**}$, a second dual space.

It was proved by Haydon [13] that every injective space isomorphic to a second dual is isomorphic to $\ell_\infty(I)$ for some set I .

- (2) Let μ be a finite measure for which $L_1(\mu)$ non-separable. Then $L_\infty(\mu) \equiv L_1(\mu)^*$ is 1-injective, but it is not isomorphic to a second dual space.
- (3) Rosenthal [23] proved that there exists a Stonian compact K_G such that $C(K_G)$ is not isomorphic to any dual space.

The following result of Rosenthal is helpful to show that some Banach spaces are not injective.

Proposition 4. [24]

(a) Every infinite dimensional injective Banach space contains a subspace isomorphic to ℓ_∞ .

(b) If an injective space contains a subspace isomorphic to $c_0(I)$, then it also contains a subspace isomorphic to $\ell_\infty(I)$.

Corollary 5. The quotient space ℓ_∞/c_0 is not injective.

Proof. Let $\{A_i : i \in I\}$ be an uncountable family of infinite subsets of \mathbb{N} such that $A_i \cap A_j$ is finite for $i \neq j$. The characteristic function of each A_i corresponds to an element $x_i \in \ell_\infty$. Let z_i denote the image of x_i in ℓ_∞/c_0 .

The subspace generated by $\{z_i : i \in I\}$ in ℓ_∞/c_0 is isomorphic to $c_0(I)$. However, ℓ_∞/c_0 does not contain subspaces isomorphic to $\ell_\infty(I)$. \square

3. SEPARABLY INJECTIVE SPACES

Let X be a Banach space and let Y be a subspace of X . We say that a Banach space E satisfies the λ -extension property for (X, Y) if each operator $T: Y \rightarrow E$ has an extension $\widehat{T}: X \rightarrow E$ with $\|\widehat{T}\| \leq \lambda\|T\|$.

Definition 6. Let $1 \leq \lambda < \infty$.

E is λ -separably injective ($E \in \Upsilon_\lambda$) if it satisfies the λ -extension property for (X, Y) when X is separable.

E is λ -universally separably injective ($E \in \Upsilon_\lambda^{univ}$) if it satisfies the λ -extension property for (X, Y) when Y is separable.

Notations: $\Upsilon := \bigcup_{\lambda \geq 1} \Upsilon_\lambda$, $\Upsilon^{univ} := \bigcup_{\lambda \geq 1} \Upsilon_\lambda^{univ}$.

Proposition 7. The following implications hold:

E injective $\Rightarrow E \in \Upsilon^{univ} \Rightarrow E \in \Upsilon \Rightarrow E \in \mathcal{L}_\infty$.

All the converse implications fail, in general.

If E is isomorphic to a dual space and $E \in \mathcal{L}_\infty$ then E is injective.

3.1. Earlier results. Several people have studied separably injective Banach spaces. Here we describe some of their results.

Proposition 8. The following assertions hold:

- (1) Let I be an infinite set. Then $c_0(I) \in \Upsilon_2$ (Sobczyk, [25]). However, $c_0(I)$ is not universally separably injective.
- (2) If $E \in \Upsilon$ is infinite dimensional and separable then E is isomorphic to c_0 (Zippin, [26]).
- (3) If E is infinite dimensional and $E \in \Upsilon_\lambda$ with $\lambda < 2$ then E is non-separable (Ostrovskii, [22]).

Next we give a good description of the $C(K)$ spaces which are 1-separably injective due to several authors (see [5]). Recall that a compact space is a F -space if disjoint open F_σ subsets have disjoint closures.

Theorem 9. For a compact space K , the following assertions are equivalent:

- (a) $C(K)$ is 1-separably injective;
- (b) given (f_i) and (g_j) in $C(K)$ with $f_i \leq g_j$ for each i, j there exists $h \in C(K)$ such that $f_i \leq h \leq g_j$ for each i, j ;
- (c) Every sequence of mutually intersecting balls in $C(K)$ has nonempty intersection;
- (d) K is a F -space;
- (e) Given $f \in C(K)$ there is $u \in C(K)$ such that $f = u|f|$.

Observe that part (e) in the previous Theorem has several applications:

- (1) A closed subset of a compact F -space is a F -space; in particular, $\ell_\infty/c_0 \equiv C(\beta\mathbb{N} \setminus \mathbb{N}) \in \Upsilon_1$.
- (2) The space $B[0, 1]$ of bounded Borel functions on $[0, 1]$ belongs to Υ_1 .

3.2. Characterizations and properties. We present several characterizations of the separably injective Banach spaces and describe some stability properties of the class Υ that allow us to obtain new examples from the previously known ones.

Proposition 10. *For a Banach space E , the following assertions are equivalent:*

- (1) $E \in \Upsilon$;
- (2) if X/Y is separable, every $T : Y \rightarrow E$ extends to X ;
- (3) $X \supset M \simeq E$, X/M separable $\Rightarrow M$ complemented in X ;
- (4) if $Y \subset \ell_1$, every $T : Y \rightarrow E$ extends to ℓ_1 .

Proposition 11. *Let $E \in \Upsilon_\lambda$ infinite dimensional. Then:*

- (i) E is a $\mathcal{L}_{\infty, \lambda}$ -space;
- (ii) E contains a copy of c_0 ;
- (iii) E has Pełczyński's property (V): every non-weakly compact $T : E \rightarrow Y$ is an isomorphism on a subspace of E isomorphic to c_0 .

We say that a class \mathcal{C} of Banach spaces has the *three-space property* if the following condition is satisfied:

$$Y \subset X; \quad Y, X/Y \in \mathcal{C} \Rightarrow X \in \mathcal{C}.$$

We refer to [7] for information on classes of Banach spaces with the three-space property. The following properties allow us to construct examples of separably injective Banach spaces.

Proposition 12.

- (i) The class Υ has the three-space property;
- (ii) $X \supset M$, $X, M \in \Upsilon \Rightarrow X/M \in \Upsilon$;
- (iii) $(E_n) \subset \Upsilon_\lambda \Rightarrow c_0(E_n) \in \Upsilon_{\lambda(1+\lambda)}$.

3.3. On universally separably injective spaces. First we describe a natural example of universally separably injective Banach space.

Let Γ be an uncountable set. We denote

$$\ell_\infty^c(\Gamma) := \{(a_i) \in \ell_\infty(\Gamma) : \text{supp } ((a_i)) \text{ countable}\}.$$

Proposition 13. $\ell_\infty^c(\Gamma) \in \Upsilon_1^{univ}$, but it is not injective.

Proof. Given an infinite countable subset J of Γ , $\{(a_i) \in \ell_\infty^c(\Gamma) : \text{supp } ((a_i)) \subset J\}$ is a subspace of $\ell_\infty^c(\Gamma)$ isometric to ℓ_∞ . Therefore, every separable subspace of $\ell_\infty^c(\Gamma)$ is contained in a subspace isometric to ℓ_∞ . From this fact, it follows that $\ell_\infty^c(\Gamma) \in \Upsilon_1^{univ}$.

The space $\ell_\infty^c(\Gamma)$ is not injective because it contains a subspace isomorphic to $c_0(\Gamma)$, but it does not contain subspaces isomorphic to $\ell_\infty(\Gamma)$. \square

Surprisingly, the property that allowed us to show that $\ell_\infty^c(\Gamma) \in \Upsilon^{univ}$ characterizes the universally separably injective spaces.

Theorem 14 (Structure). $E \in \Upsilon^{univ}$ if and only if each separable subspace of E is contained in another subspace isomorphic to ℓ_∞ .

The following result shows that infinite dimensional spaces in Υ^{univ} are big.

Proposition 15. If $E \in \Upsilon^{univ}$, every non-weakly compact operator $T : E \rightarrow Y$ is an isomorphism on a subspace of E isomorphic to ℓ_∞ .

Recall that two Banach spaces E and F are *essentially incomparable* if given operators $T : E \rightarrow F$ and $S : F \rightarrow E$, $I_E - ST$ (equivalently, $I_F - TS$) is a bijective isomorphism up to some finite dimensional subspaces [8].

Corollary 16. *Every infinite dimensional space in Υ^{univ} contains a subspace isomorphic to ℓ_∞ .*

If $E \in \Upsilon^{univ}$ and F contains no subspaces isomorphic to ℓ_∞ then E and F are essentially incomparable.

The following result describe some stability properties of the class Υ^{univ}

Proposition 17 (Construction of examples).

(i) *The class Υ^{univ} has the three-space property:*

$$Y \subset X; \quad Y, X/Y \in \Upsilon^{univ} \Rightarrow X \in \Upsilon^{univ};$$

(ii) *$X \supset M$, $X \in \Upsilon^{univ}$, $M \in \Upsilon \Rightarrow X/M \in \Upsilon^{univ}$.*

3.4. Special properties of spaces in Υ_1 . Here we present some properties of the 1-separably injective Banach spaces in which special axioms of set theory are involved.

We denote by C.H. the *continuum hypothesis*: $\mathfrak{c} = \aleph_1$, and Z.F.C. represents the *Zermelo-Fraenkel axioms*, including Choice.

Proposition 18. *Let E be a 1-separably injective space. Then*

- (1) *E is Grothendieck; i.e., every operator from E into c_0 is weakly compact;*
- (2) *E is a Lindenstrauss space; i.e., E^* is linearly isometric to some $L_1(\mu)$ space;*
- (3) *if E is infinite dimensional, then $\text{dens}(E) \geq \mathfrak{c}$.*

The following result is a direct application of an argument of Lindenstrauss [17].

Proposition 19. *Under C.H., the classes Υ_1 and Υ_1^{univ} coincide.*

Corollary 20. *Under C.H., every $E \in \Upsilon_1$ contains a subspace isomorphic to ℓ_∞ .*

In the following result we show that C.H. is necessary for the coincidence of Υ_1 and Υ_1^{univ} .

Theorem 21. *Under Z.F.C. + $\mathfrak{c} = \aleph_2$, there exists a compact space K_0 such that $C(K_0) \in \Upsilon_1$ but $C(K_0) \notin \Upsilon_1^{univ}$.*

We observe that we do not know if the space $C(K_0)$ in the previous Theorem belongs to Υ^{univ} .

Let Ku denote the Banach space of universal disposition for separable spaces constructed by Kubis [16]. Observe that Ku is not isomorphic to any $C(K)$ space.

Proposition 22. *Under C.H., $Ku \in \Upsilon_1^{univ}$.*

3.5. Ideals and M -ideals. Here we give some results for closed ideals of $C(K)$ spaces and the corresponding quotients. We also give some related abstract results in terms of M -ideals in Banach spaces.

Let M be a closed subset of a compact K . Then $L := K \setminus M$ is locally compact. Moreover, $C_0(L)$ is a closed ideal in $C(K)$ and the quotient space $C(K)/C_0(L)$ is isometric to $C(M)$. Consequently, we have an exact sequence

$$0 \longrightarrow C_0(L) \longrightarrow C(K) \longrightarrow C(M) \longrightarrow 0.$$

Theorem 23. *Let M be a closed subset of a compact K .*

- (1) $C(K) \in \Upsilon_\lambda^{univ} \Rightarrow C(M) \in \Upsilon_\lambda^{univ}$;
- (2) $C(K) \in \Upsilon_\lambda \Rightarrow C_0(L) \in \Upsilon_{2\lambda}$.

Recall that a closed subspace J of a Banach space E is a M -ideal if $E^* = J^\perp \oplus_1 N$ for some closed subspace N .

Theorem 24. *Let J be a M -ideal in E .*

- (1) $E \in \Upsilon_\lambda^{univ} \Rightarrow E/J \in \Upsilon_{\lambda^2}^{univ}$;
- (2) $E \in \Upsilon_\lambda \Rightarrow J \in \Upsilon_{2\lambda^2}$.

3.6. Ultraproducts of Banach spaces. Here we give some results involving ultraproducts of Banach spaces. First, we recall the concept of ultraproduct. For additional information, we refer to [14] or [9, A4].

Let I be an infinite set and let \mathcal{U} be a countably incomplete ultrafilter on I . Recall that \mathcal{U} is countably incomplete if and only if there exists a sequence (I_n) of subsets of I in \mathcal{U} such that $\bigcap_{n=1}^\infty I_n = \emptyset$.

Let $(X_i)_{i \in I}$ be a family of Banach spaces. Then

$$\ell_\infty(X_i) := \{(x_i) : x_i \in X_i, \sup_i \|x_i\| < \infty\},$$

endowed with the supremum norm, is a Banach space, and

$$c_0^\mathcal{U}(X_i) := \{(x_i) \in \ell_\infty(X_i) : \lim_{i \rightarrow \mathcal{U}} \|x_i\| = 0\}$$

is a closed subspace of $\ell_\infty(X_i)$.

The *ultraproduct* of $(X_i)_{i \in I}$ following \mathcal{U} is defined as the quotient

$$(X_i)_\mathcal{U} := \frac{\ell_\infty(X_i)}{c_0^\mathcal{U}(X_i)}.$$

If $[x_i]$ is the element of $(X_i)_\mathcal{U}$ which has (x_i) as a representative then

$$\|[x_i]\| = \lim_{i \rightarrow \mathcal{U}} \|x_i\|.$$

In the case $X_i = X$ for all i , we denote the ultraproduct by $X_\mathcal{U}$, and call it the *ultrapower* of X following \mathcal{U} .

3.7. Ultraproducts which are \mathcal{L}_∞ -spaces. Recall that X is a \mathcal{L}_∞ space if there exists λ ($1 \leq \lambda < \infty$) such that every finite dimensional subspace of X is contained in another finite dimensional subspace F so that $\text{dist}_{BM}(F, \ell_\infty^{\dim F}) \leq \lambda$. We refer to [6, 19] for information on \mathcal{L}_∞ -spaces.

Our first result says that non-trivial ultraproducts are never injective.

Theorem 25. *$(X_i)_\mathcal{U}$ is injective if and only if it is finite dimensional.*

Our second results says that ultraproducts which are \mathcal{L}_∞ -spaces belong to Υ^{univ} .

Theorem 26. *$(X_i)_\mathcal{U}$ \mathcal{L}_∞ -space $\Rightarrow (X_i)_\mathcal{U} \in \Upsilon^{univ}$.*

Let us state some consequences:

- (1) $(X_i) \subset \mathcal{L}_{\infty, \lambda+} \Rightarrow (X_i)_\mathcal{U} \in \Upsilon_\lambda^{univ}$.
- (2) (X_i) Lindenstrauss spaces (e.g., $C(K)$ spaces) $\Rightarrow (X_i)_\mathcal{U} \in \Upsilon_1^{univ}$.

Let $\mathcal{G}u$ denote the Banach space of universal disposition for finite dimensional spaces constructed by Gurarii [11].

Theorem 27. *The ultrapower $(\mathcal{G}u)_{\mathcal{U}}$ belongs to Υ_1^{univ} , but it is not isomorphic to a complemented subspace of any $C(K)$ space.*

It was proved by Kubis [16] that, under C.H., there is only one Banach space of universal disposition for separable spaces with density character \aleph_1 . As a consequence, we derive the following result.

Proposition 28. *Under C.H., for each non-trivial ultrafilter \mathcal{U} on \mathbb{N} , $\mathcal{K}u = (\mathcal{G}u)_{\mathcal{U}}$.*

3.8. Automorphic character. Let \mathcal{C} be a class of Banach spaces.

We say that a Banach space E is *automorphic for \mathcal{C}* if, given subspaces M_1 and M_2 of E with $M_1 \simeq M_2 \in \mathcal{C}$ and $\text{dens}(E/M_1) = \text{dens}(E/M_2) \geq \aleph_0$, each bijective isomorphism $j : M_1 \rightarrow M_2$ extends to an automorphism of E .

The following list contains all known examples of Banach spaces which are automorphic for all their subspaces:

- (1) $\ell_2(I)$ (trivial);
- (2) c_0 (Lindenstrauss-Rosenthal [18]);
- (3) $c_0(I)$ (Moreno-Plichko [20]).

Remark 29. *It would be interesting to know if c_0 and ℓ_2 are the only infinite dimensional separable spaces which are automorphic for all their subspaces.*

Let us see the relations between the automorphic character of a space and its extension properties.

Proposition 30. *Let E be a Banach space automorphic for separable spaces.*

- (1) *If E contains a subspace isomorphic to ℓ_1 then $E \in \Upsilon$.*
- (2) *If E contains a subspace isomorphic to ℓ_∞ then $E \in \Upsilon^{univ}$.*

3.9. Automorphic character of Banach spaces in Υ^{univ} . Recall that an operator $U : X \rightarrow Y$ is *Fredholm* if the kernel $\ker(U)$ and the cokernel $Y/U(X)$ are finite dimensional (hence $U(X)$ is closed). In this case, we define the *index* of U by

$$\text{ind}(U) := \dim \ker(U) - \dim Y/U(X).$$

The following two Propositions were proved by Lindenstrauss and Rosenthal [18] for $E = \ell_\infty$.

Proposition 31. *Let M be a subspace of $E \in \Upsilon^{univ}$.*

If $j : M \rightarrow E$ is an isomorphism and E/M and $E/j(M)$ are reflexive, then there are extensions $U : E \rightarrow E$ of j .

All the extensions are Fredholm operators with the same index.

Proposition 32. *Each $E \in \Upsilon^{univ}$ is automorphic for separable spaces.*

We say that a subspace M of a Banach space E is $c_0(I)$ -*supplemented* if there exists another subspace N of E isomorphic to $c_0(I)$ such that $M \cap N = \{0\}$ and $M + N$ closed.

Proposition 33. *Every $E \in \Upsilon^{univ}$ is automorphic for subspaces of $\ell_\infty(I)$ which are $c_0(I)$ -supplemented; i.e., if M_1 and M_2 are $c_0(I)$ -supplemented subspaces of E and $M_1 \simeq M_2 \simeq N \subset \ell_\infty(I)$, then each isomorphism from M_1 onto M_2 extends to an automorphism of E .*

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4. OPEN PROBLEMS

Here we describe some questions which remain unsolved.

- (1) $X, Y \in \Upsilon \Rightarrow X \hat{\otimes}_\varepsilon Y \in \Upsilon$? ($\hat{\otimes}_\varepsilon$: injective tensor product)

We have a positive answer to (1) in a special case: $Y \in \Upsilon \Rightarrow c_0(Y) \equiv c_0 \hat{\otimes}_\varepsilon Y \in \Upsilon$.

The corresponding implication for Υ^{univ} fails because $X \supset c_0$ and $\dim Y = \infty$ imply $X \hat{\otimes}_\varepsilon Y \supset c_0$ complemented.

We do not know the answer in the case $X = Y = \ell_\infty$.

- (2) Characterize the compact spaces K for which $C(K) \in \Upsilon^{univ}$.

We conjecture that K σ -Stonian $\Rightarrow C(K) \in \Upsilon^{univ}$, where K is σ -Stonian if the closure of each open F_σ -set is open.

- (3) Is $\ell_\infty/C[0, 1]$ separably injective?

Note that, since ℓ_∞ is automorphic for separable spaces, the quotient $\ell_\infty/C[0, 1]$ does not depend on the way we embed $C[0, 1]$ into ℓ_∞ .

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